4-th Italian Mathematical Olympiad 1988

Viareggio, April 25, 1988

1. Players A and B play the following game: A tosses a coin n times, and B does $n+1$ times. The player who obtains more "heads" wins; or in the case of equal balances, A is assigned victory. Find the values of n for which this game is fair (i.e. both players have equal chances for victory).

2. In a basketball tournament any two of the n teams $S_1, S_2, \dots, S_n$ play one match (no draws). Denote by v_i and p_i the number of victories and defeats of team S_i ($i = 1, 2, \dots, n$), respectively. Prove that
$$v_1^2 + v_2^2 + \cdots + v_n^2 = p_1^2 + p_2^2 + \cdots + p_n^2.$$

3. A regular pentagon of side length 1 is given. Determine the smallest r for which the pentagon can be covered by five discs of radius r and justify your answer.

4. Show that all terms of the sequence $1, 11, 111, 1111, \dots$ in base 9 are triangular numbers, i.e. of the form $\frac{m(m+1)}{2}$ for an integer m.

5. Given four non-coplanar points, is it always possible to find a plane such that the orthogonal projections of the points onto the plane are the vertices of a parallelogram? How many such planes are there in general?

6. The edge lengths of the base of a tetrahedron are a, b, c, and the lateral edge lengths are x, y, z. If d is the distance from the top vertex to the centroid of the base, prove that
$$x + y + z \le a + b + c + 3d.$$

7. Given $n \ge 3$ positive integers not exceeding 100, let d be their greatest common divisor. Show that there exist three of these numbers whose greatest common divisor is also equal to d.

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